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EINSTEIN SPACES OF POSITIVE SCALAR CURVATURE

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1. Let M be an n-dimensional compact orientable Einstein space with positive scalar curvature K. Then the concircular curvature tensor Z_{kjih} defined by

$$Z_{kjih} = K_{kjih} - \frac{K}{n(n-1)}(g_{ji}g_{kh} - g_{ki}g_{jh}) \tag{1}$$

satisfies

$$Z_{kjih}g^{ji} = 0 \,, \tag{2}$$

because of

$$K_{ji} = \frac{K}{n}g_{ji} \,.$$

The purpose of the present paper is to prove the

Theorem. *If the concircular curvature tensor satisfies the inequality*

$$\frac{1}{K}|Z_{kjih}A^kB^jC^iD^h| < \frac{2}{5n^7} \tag{3}$$

at every point of M for any set of unit vectors A, B, C, D, then the Einstein space M is a space of constant curvature.

Roughly speaking, this theorem tells that, if M_0 is a space of positive constant curvature, there exist no Einstein spaces other than M_0 in a sufficiently small neighborhood of M_0. The inequality (3) is a quite rough and modest estimation. We can get a better estimation by a more elaborate calculation.

2. In an Einstein space we have $\nabla_k K_{ji} = 0$, $\nabla_k K = 0$, and therefore $\nabla_l Z_{kjih} = \nabla_l K_{kjih}$. Thus by using Green's theorem and the second identity of Bianchi, we get

$$\begin{aligned} -\int_M (\nabla^l Z^{kjih})(\nabla_l Z_{kjih})dV &= \int_M Z^{kjih}\nabla^l\nabla_l K_{kjih}dV \\ &= 2\int_M Z^{kjih}\nabla^l\nabla_k K_{ljih}dV \,, \end{aligned}$$

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where dV is the volume element of M, and the last step becomes, by virtue of the Ricci identity and $\nabla^l K_{ljih} = 0$,

$$2\int_M Z^{kjih}\left(\frac{K}{n}K_{kjih} - K^l{}_{kj}{}^m K_{lmih} - K^l{}_{ki}{}^m K_{ljmh} - K^l{}_{kh}{}^m K_{ljim}\right)dV$$

$$= \frac{2K}{n}\int_M Z^{kjih}K_{kjih}dV + \int_M Z^{kjih}(K_{kj}{}^{lm}K_{lmih} - 4K^l{}_{kh}{}^m K_{jlmi})dV\,,$$

where we have used

$$K^l{}_{kj}{}^m - K^l{}_{jk}{}^m = -K_{kj}{}^{lm}\,.$$

We also obtain, in consequence of (1),

$$\int_M Z^{kjih}K_{kjih}dV = \int_M Z^{kjih}Z_{kjih}dV\,,$$

$$\int_M Z^{kjih}(K_{kj}{}^{lm}K_{lmih} - 4K^l{}_{kh}{}^m K_{jlmi})dV$$

$$= \int_M (Z^{kjih}Z_{kj}{}^{lm}Z_{lmih} - 4Z^{kjih}Z^l{}_{kh}{}^m Z_{jlmi})dV$$

$$+ \frac{K}{n(n-1)}\int_M Z^{kjih}(Z_{kjhi} - Z_{kjih} + Z_{jkih} - Z_{kjih})dV$$

$$+ \frac{4K}{n(n-1)}\int_M Z^{kjih}(Z_{ikhj} + Z_{jhki})dV\,.$$

In the last step the second and the third terms are cancelled with each other by virtue of the first identity of Bianchi, so that we have

$$(4)\qquad \begin{aligned} &-\int_M (\nabla^l Z^{kjih})(\nabla_l Z_{kjih})dV \\ &= \int_M \left(\frac{2K}{n}Z^{kjih}Z_{kjih} + Z^{kjih}Z_{kj}{}^{lm}Z_{lmih} - 4Z^{kjih}Z^l{}_{kh}{}^m Z_{jlmi}\right)dV\,. \end{aligned}$$

3. At each point P of M let us take an orthonormal frame and consider all components of Z_{kjih} with respect to this frame. By defining $m(P)$ by

$$m(P) = \max\left(\frac{1}{K}|Z_{kjih}|\right),$$

we obtain

$$\int_M Z^{kjih} Z_{kjih} dV \geq K^2 \int_M m^2 dV ,$$

$$\int_M Z^{kjih} Z_{kj}{}^{lm} Z_{lmih} dV \geq -n^6 K^3 \int_M m^3 dV ,$$

$$-\int_M Z^{kjih} Z^l{}_{kh}{}^m Z_{jlmi} dV \geq -n^6 K^3 \int_M m^3 dV .$$

Hence we have the inequality

$$(5) \qquad \int_M m^2 \left(1 - \frac{5}{2} n^7 m\right) dV \leq 0 .$$

If (3) holds, then m satisfies

$$m < \frac{2}{5n^7}$$

on M, and we have

$$\int_M m^2 \left(1 - \frac{5}{2} n^7 m\right) dV \geq 0 ,$$

which and (5) imply that in this case m must be identically zero, and therefore

$$Z_{kjih} = 0 .$$

Hence the theorem is proved.

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